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Anyonic realization of the quantum affine Lie superalgebras $\mathcal{U}_q(\widehat{A}(M-1, N-1))$

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Abstract. We give a realization of the quantum affine Lie superalgebras $\mathcal{U}_q(\widehat{A}(M-1, N-1))$ in terms of anyons defined on a one- or two-dimensional lattice, the deformation parameter q being related to the statistical parameter ν of the anyons by $q = e^{i\pi\nu}$. The construction uses anyons contructed from usual fermionic oscillators and deformed bosonic oscillators. As a byproduct, a realization that is deformed in any sector of the quantum superalgebras $\mathcal{U}_q(A(M-1, N-1))$ is obtained.

1. Introduction

Superalgebras, which are the mathematical framework for describing symmetry betwen bosons and fermions, have by now found several interesting applications in physics, even if the *fundamental* supersymmetry between elementary constituents of matter has not yet been supported by experimental evidence. A further enlarged concept of symmetry represented by quantum algebras has shown up in a large number of areas in physics.The fusion of these new enlarged symmetry structures is a natural step, and it leads to the so-called q-superalgebras.

Moreover, the connection between quantum algebras and generalized statistics has been pointed out in several contexts. *Anyons* are typical objects of generalized statistics whose importance in two-dimensional physics is relevant. They have been used to construct a Schwinger–Jordan-like realization of the deformed classical finite Lie algebras [7, 6] and of the deformed affine Lie algebras of the unitary and symplectic series [3, 4]. So it is natural to ask which kind of oscillators are necessary to build realizations of q-superalgebras. We will show that the deformation of the affine unitary superalgebras (and therefore also of the finite unitary superalgebras) can be realized by means of anyons and of a new type of generalized statistical object which satisfy braiding relations and which will be called *bosonic anyons* for reasons which will be clear from their definitions (see section 4).

Let us emphasize that the construction we propose may be interesting in the study of systems of correlated electrons. In fact the so-called t–J model [14], which has been suggested as an appropriate starting point for the theory of the high-temperature

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superconductivity, is supersymmetric for particular values of the coupling constants and of the chemical potential, the Hamiltonian commuting with a $su(1|2)$. Moreover, in [8] it was shown that the t–J model at the supersymmetric point can be written in terms of anyons, which gives a new realization of supersymmetry. Although no deformation appears in this model, it is conceivable that anyons can be used to describe further deformed generalizations of this or of similar models, like the Hubbard model [10].

The article is organized as follows. In section 2 we briefly recall the structure of $\widehat{A}(M-1,N-1)$ in the Cartan–Weyl and Serre–Chevalley bases and then we write its deformation in the distinguished Serre–Chevalley basis. In section 3 the fermionic-bosonic oscillators realization of $\widehat{A}(M-1,N-1)$ is presented and finally in section 4 the anyonic realization of $\mathcal{U}_q(\widehat{A}(M-1,N-1))$ is given in terms of one-dimensional anyons and bosonic anyons. In section 5 the generalization of the construction to two-dimensional anyons is discussed and a few conclusions are presented.

2. Presentation of the superalgebra $\widehat{A}(M-1,N-1)$

We will recall in this section the presentation of the affine Lie superalgebra $\widehat{A}(M-1,N-1)$, where $M,N \geqslant 1$, both in the Cartan–Weyl basis and in the Serre–Chevalley basis. We set $R = M+N-1$ and we exclude the case $R = 1$ (obtained when $M = N = 1$).

2.1. *Cartan–Weyl presentation of $\widehat{A}(M-1,N-1)$*

In the Cartan–Weyl basis, the generators of the affine Lie superalgebra $\widehat{A}(M-1,N-1)$ are denoted by h_a^m (Cartan generators) and $e_{\mathfrak{a}}^m$ (root generators) where $a = 1,\dots,R$ and $m \in \mathbb{Z}$. The even root system of $\widehat{A}(M-1,N-1)$ is given by $\Delta_0 = \{\pm(\varepsilon_i-\varepsilon_j), \pm(\delta_k-\delta_l)\}$ and the odd root system by $\Delta_1 = \{\pm(\varepsilon_i-\delta_k)\}$ where $1 \leqslant i < j \leqslant M$ and $1 \leqslant k < l \leqslant N$, the ε_i and δ_k spanning the dual of the Cartan subalgebra of $gl(M|N)$. To each root generator $e_{\mathfrak{a}}^m$ one assigns a $\mathbb{Z}_2$-grading defined by $\deg(e_{\mathfrak{a}}^m) = 0$ if $\mathfrak{a} \in \Delta_0$ and $\deg(e_{\mathfrak{a}}^m) = 1$ if $\mathfrak{a} \in \Delta_1$. The generators satisfy the following commutation relations for $M \neq N$:

$$\left[h_a^m, h_b^n\right] = \gamma\, m\, \delta_{m+n,0}\, K(h_a,h_b) \tag{2.1a}$$

$$\left[h_a^m, e_{\mathfrak{a}}^n\right] = \mathfrak{a}_a\, e_{\mathfrak{a}}^{m+n} \tag{2.1b}$$

$$[\![e_{\mathfrak{a}}^m, e_{\mathfrak{b}}^n]\!] = \begin{cases} \varepsilon(\mathfrak{a},\mathfrak{b})\, e_{\mathfrak{a}+\mathfrak{b}}^{m+n} & \text{if } \mathfrak{a}+\mathfrak{b} \text{ is a root} \\ \mathfrak{a}^a h_a^{m+n} + \gamma\, m\, \delta_{m+n,0}\, K(e_{\mathfrak{a}}, e_{-\mathfrak{a}}) & \text{if } \mathfrak{b} = -\mathfrak{a} \\ 0 & \text{otherwise} \end{cases} \tag{2.1c}$$

$$\left[h_a^m, \gamma\right] = \left[e_{\mathfrak{a}}^m, \gamma\right] = 0 \tag{2.1d}$$

where $\varepsilon(\mathfrak{a},\mathfrak{b}) = \pm 1$ is the usual 2-cocycle, K is the (non-degenerate) Killing form on the horizontal superalgebra $A(M-1,N-1)$ and γ is the central charge. $[\![\,,\,]\!]$ denotes the super-commutator: $[\![e_{\mathfrak{a}}^m, e_{\mathfrak{b}}^n]\!] = e_{\mathfrak{a}}^m\, e_{\mathfrak{b}}^n - (-1)^{\deg(e_{\mathfrak{a}}^m)\cdot\deg(e_{\mathfrak{b}}^n)} e_{\mathfrak{b}}^n\, e_{\mathfrak{a}}^m$. Note that by virtue of equations (2.1a)–(2.1d) the value of the central charge of $\hat{A}_{M-1}$ is opposite to that of $\hat{A}_{N-1}$.

In the case $M = N$, although the Killing form is zero, it is possible to define a non-degenerate bilinear form K on $A(N-1,N-1)$ such that (2.1a)–(2.1d) still hold.

2.2. *Serre–Chevalley presentation of* $\widehat{A}(M-1\,,\, N-1)$

In the Serre–Chevalley basis, the algebra is described in terms of simple root and Cartan generators, the only data being the entries of the Cartan matrix $(a_{\alpha\beta})$ of the algebra. Let us denote the generators in the Serre–Chevalley basis by h_α and $e^{\pm}_{\alpha}$ where $\alpha = 0, 1, \ldots, R$. If τ is a subset of $\{0, 1, \ldots, R\}$, the $\mathbb{Z}_2$-gradation of the superalgebra is defined by setting $\deg(e^{\pm}_{\alpha}) = 0$ if $\alpha \notin \tau$ and $\deg(e^{\pm}_{\alpha}) = 1$ if $\alpha \in \tau$. The superalgebra is described by the (super)commutation relations

$$\left[h_\alpha, h_\beta\right] = 0 \tag{2.2a}$$

$$\left[h_\alpha, e^{\pm}_{\beta}\right] = \pm a_{\alpha\beta} e^{\pm}_{\beta} \tag{2.2b}$$

$$\left[\!\left[e^{+}_{\alpha}, e^{-}_{\beta}\right]\!\right] = e^{+}_{\alpha} e^{-}_{\beta} - (-1)^{\deg(e^{+}_{\alpha})\deg(e^{-}_{\beta})}\, e^{-}_{\beta} e^{+}_{\alpha} = h_\alpha\, \delta_{\alpha\beta} \tag{2.2c}$$

$$\left\{e^{\pm}_{\alpha}, e^{\pm}_{\alpha}\right\} = 0 \qquad \text{if} \;\; a_{\alpha\alpha} = 0 \tag{2.2d}$$

and by the following relations:

- the Serre relations for all $\alpha \neq \beta$

$$(\mathrm{ad}\, e^{\pm}_{\alpha})^{1-\tilde{a}_{\alpha\beta}}\, e^{\pm}_{\beta} = 0 \tag{2.3}$$

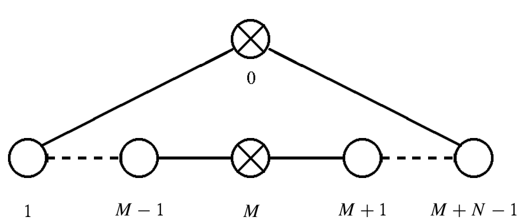
- supplementary relations for α such that $a_{\alpha\alpha} = 0$

$$\left[\!\left[(\mathrm{ad}\, e^{\pm}_{\alpha-1})\, e^{\pm}_{\alpha}, (\mathrm{ad}\, e^{\pm}_{\alpha+1})\, e^{\pm}_{\alpha}\right]\!\right] = 0 \tag{2.4}$$

where the matrix $\tilde{A} = (\tilde{a}_{\alpha\beta})$ is deduced from the Cartan matrix $A = (a_{\alpha\beta})$ of $\widehat{A}(M-1, N-1)$ by replacing all its positive off-diagonal entries by -1. Here ad denotes the adjoint action: $(\mathrm{ad}\, X)\, Y = XY - (-1)^{\deg X \cdot \deg Y}\, YX$.

One has to emphasize that for superalgebras, the description given by the Serre relations (2.3) leads in general to a bigger superalgebra than the superalgebra under consideration. It is necessary to write supplementary relations involving more than two generators, that for $A(M-1, N-1)$ take the form (2.4), in order to quotient the bigger superalgebra and recover the original one; see [11] for more details. As one can imagine, these supplementary conditions appear when one deals with isotropic fermionic simple roots (that is $a_{\alpha\alpha} = 0$). Note that these supplementary relations are unnecessary when $M = 1$ or $N = 1$.

In what follows, we will only use the Serre–Chevalley description of the affine Lie superalgebra in the *distinguished* basis, such that the number of odd simple roots is the smallest one. In the case of $\widehat{A}(M-1, N-1)$, the distinguished basis is defined by taking $\tau = \{0, M\}$. The corresponding Dynkin diagram is as follows, with the labels identifying the corresponding simple roots:



associated with the Cartan matrix

$$
\begin{pmatrix}
0 & 1 & 0 & \cdots & \cdots & & & & \cdots & \cdots & 0 & -1 \\
-1 & 2 & -1 & 0 & & & & & & & & 0 \\
0 & -1 & \ddots & \ddots & \ddots & & & & & & & \vdots \\
\vdots & 0 & \ddots & & \ddots & 0 & & & & & & \vdots \\
\vdots & & \ddots & \ddots & \ddots & -1 & \ddots & & & & & \\
 & & & 0 & -1 & 2 & -1 & \ddots & & & & \\
 & & & & \ddots & -1 & 0 & 1 & \ddots & & & \\
 & & & & & \ddots & -1 & 2 & -1 & 0 & & \vdots \\
\vdots & & & & & & \ddots & -1 & \ddots & \ddots & \ddots & \vdots \\
\vdots & & & & & & & 0 & \ddots & & \ddots & 0 \\
0 & & & & & & & & \ddots & \ddots & \ddots & -1 \\
-1 & 0 & \cdots & \cdots & & \cdots & \cdots & 0 & \cdots & 0 & -1 & 2
\end{pmatrix}. \qquad (2.5)
$$

The correspondence between the distinguished Serre–Chevalley and the Cartan–Weyl bases is as follows ($i = 1, \ldots, M-1$ and $k = 1, \ldots, N-1$):

$$
\begin{aligned}
&h_i = h_i^0 && e_i^+ = e^0_{\varepsilon_i - \varepsilon_{i+1}} && e_i^- = e^0_{\varepsilon_{i+1} - \varepsilon_i} \\
&h_M = h_M^0 && e_M^+ = e^0_{\varepsilon_M - \delta_1} && e_M^- = e^0_{\delta_1 - \varepsilon_M} \\
&h_{M+k} = h_{M+k}^0 && e_{M+k}^+ = e^0_{\delta_k - \delta_{k+1}} && e_{M+k}^- = e^0_{\delta_{k+1} - \delta_k} \\
&h_0 = -\gamma + \sum_{i=1}^{M} h_i^0 - \sum_{k=1}^{N-1} h_{M+k}^0 && e_0^+ = e^1_{\delta_N - \varepsilon_1} && e_0^- = e^{-1}_{\varepsilon_1 - \delta_N}.
\end{aligned} \qquad (2.6)
$$

Note that in the Serre–Chevalley picture the central charge γ is uniquely defined by the following equation:

$$
h_0 = -\gamma + \sum_{i=1}^{M} h_i - \sum_{k=1}^{N-1} h_{M+k}. \qquad (2.7)
$$

2.3. Serre–Chevalley presentation of $\mathcal{U}_q(\widehat{A}(M-1, N-1))$

We now consider the universal quantum affine Lie superalgebra $\mathcal{U}_q(\widehat{A}(M-1, N-1))$. The Serre–Chevalley description in the quantum case is very similar. The defining relations take the form

$$
\left[h_\alpha, h_\beta\right] = 0 \qquad (2.8a)
$$

$$
\left[h_\alpha, e_\beta^\pm\right] = \pm a_{\alpha\beta} e_\beta^\pm \qquad (2.8b)
$$

$$
[\![e_\alpha^+, e_\beta^-]\!] = e_\alpha^+ e_\beta^- - (-1)^{\deg(e_\alpha^+)\deg(e_\beta^-)} e_\beta^- e_\alpha^+ = \frac{q_\alpha^{h_\alpha} - q_\alpha^{-h_\alpha}}{q_\alpha - q_\alpha^{-1}} \delta_{\alpha\beta} \qquad (2.8c)
$$

$$
\left\{e_\alpha^\pm, e_\alpha^\pm\right\} = 0 \qquad \text{if} \quad a_{\alpha\alpha} = 0 \qquad (2.8d)
$$

where $q_\alpha = q^{d_\alpha}$ and the numbers d_α symmetrize the Cartan matrix $\bar{a}_{\alpha\beta}$ of $A(M-1, N-1)$: $d_\alpha \bar{a}_{\alpha\beta} = d_\beta \bar{a}_{\beta\alpha}$ $(\alpha, \beta \neq 0)$ and $d_0 = 1$ (in the distinguished basis, $q_\alpha = q$ for $\alpha = 0, \ldots, M$ and $q_\alpha = q^{-1}$ for $\alpha = M+1, \ldots, M+N-1$).

In terms of the generators $\mathcal{E}_\alpha^\pm = e_\alpha^\pm\, q_\alpha^{-h_\alpha/2}$ the usual Serre relations are for all $\alpha \neq \beta$

$$(\mathrm{ad}_q\, \mathcal{E}_\alpha^\pm)^{1-\tilde{a}_{\alpha\beta}}\, \mathcal{E}_\beta^\pm = 0 \tag{2.9}$$

while the supplementary relations for α such that $a_{\alpha\alpha} = 0$ (the definition of the quantum adjoint action ad_q is given below (2.13)) [1, 11, 13] now read

$$\left[\!\left[(\mathrm{ad}_q\, \mathcal{E}_{\alpha-1}^\pm)\, \mathcal{E}_\alpha^\pm, (\mathrm{ad}_q\, \mathcal{E}_{\alpha+1}^\pm)\, \mathcal{E}_\alpha^\pm \right]\!\right] = 0 \tag{2.10}$$

or in terms of the generators $e_\alpha^\pm$

$$\left[\!\left[\left[e_{\alpha-1}^\pm, e_\alpha^\pm\right]_q, \left[e_\alpha^\pm, e_{\alpha+1}^\pm\right]_q \right]\!\right] = 0 \tag{2.11}$$

the q-commutator being defined as usual by $[X, Y]_q = XY - qYX$.

The universal quantum affine Lie superalgebra $\mathcal{U} \equiv \mathcal{U}_q(\widehat{A}(M-1, N-1))$ is endowed with a Hopf algebra structure, with coproduct $\Delta : \mathcal{U} \to \mathcal{U} \otimes \mathcal{U}$, counit $\varepsilon : \mathcal{U} \to \mathbb{C}$ and antipode $S : \mathcal{U} \to \mathcal{U}$ such that $(\alpha = 0, 1, \ldots, R)$

$$\Delta(h_\alpha) = 1 \otimes h_\alpha + h_\alpha \otimes 1 \qquad \text{and} \qquad \Delta(e_\alpha^\pm) = e_\alpha^\pm \otimes q_\alpha^{h_\alpha/2} + q_\alpha^{-h_\alpha/2} \otimes e_\alpha^\pm \tag{2.12a}$$

$$\varepsilon(h_\alpha) = \varepsilon(e_\alpha^\pm) = 0 \qquad \text{and} \qquad \varepsilon(1) = 1 \tag{2.12b}$$

$$S(h_\alpha) = -h_\alpha \qquad \text{and} \qquad S(e_\alpha^\pm) = -q_\alpha^{\pm a_{\alpha\alpha}/2} e_\alpha^\pm. \tag{2.12c}$$

The quantum adjoint action ad_q can be explicitly written in terms of the coproduct and the antipode as

$$(\mathrm{ad}_q\, X)\, Y = (-1)^{\deg X_{(2)} \cdot \deg Y}\, X_{(1)}\, Y\, S(X_{(2)}) \tag{2.13}$$

using the Sweedler notation for the coproduct: $\Delta(X) = X_{(1)} \otimes X_{(2)}$ (summation is understood).

3. Oscillator realization of the affine Lie superalgebra $\widehat{A}(M-1, N-1)$

Let us recall now the oscillator realization of $\widehat{A}(M-1, N-1)$ in terms of creation and annihilation operators. We consider an infinite number of fermionic oscillators $c_i(r), c_i^\dagger(r)$ with $i = 1, \ldots, M$ and $r \in \mathbb{Z} + \frac{1}{2} = \mathbb{Z}'$, which satisfy the anticommutation relations

$$\left\{c_i(r), c_j(s)\right\} = \left\{c_i^\dagger(r), c_j^\dagger(s)\right\} = 0 \qquad \text{and} \qquad \left\{c_i(r), c_j^\dagger(s)\right\} = \delta_{ij}\delta_{rs} \tag{3.1}$$

and an infinite number of bosonic oscillators $d_k(r), d_k^\dagger(r)$ with $k = 1, \ldots, N$ and $r \in \mathbb{Z}'$, which satisfy the commutation relations

$$\left[d_k(r), d_l(s)\right] = \left[d_k^\dagger(r), d_l^\dagger(s)\right] = 0 \qquad \text{and} \qquad \left[d_k(r), d_l^\dagger(s)\right] = \delta_{kl}\delta_{rs} \tag{3.2}$$

the two sets $c_i(r), c_i^\dagger(r)$ and $d_k(r), d_k^\dagger(r)$ commuting with each other:

$$\left[c_i(r), d_k(s)\right] = \left[c_i(r), d_k^\dagger(s)\right] = \left[c_i^\dagger(r), d_k(s)\right] = \left[c_i^\dagger(r), d_k^\dagger(s)\right] = 0. \tag{3.3}$$

The fermionic and bosonic number operators are defined as usual by $n_i(r) = c_i^\dagger(r)c_i(r)$ and $n'_k(r) = d_k^\dagger(r)d_k(r)$.

These oscillators are equipped with a normal ordered product such that

$$: c_i^\dagger(r)c_j(s) := \begin{cases} c_i^\dagger(r)c_j(s) & \text{if } s > 0 \\ -c_j(s)c_i^\dagger(r) & \text{if } s < 0 \end{cases} \tag{3.4}$$

and

$$: d_k^\dagger(r) d_l(s) := \begin{cases} d_k^\dagger(r) d_l(s) & \text{if } s > 0 \\ d_l(s) d_k^\dagger(r) & \text{if } s < 0. \end{cases} \tag{3.5}$$

Therefore

$$: n_i(r) := \begin{cases} n_i(r) & \text{if } r > 0 \\ n_i(r) - 1 & \text{if } r < 0 \end{cases} \tag{3.6}$$

and

$$: n'_k(r) := \begin{cases} n'_k(r) & \text{if } r > 0 \\ n'_k(r) + 1 & \text{if } r < 0. \end{cases} \tag{3.7}$$

Then an oscillator realization of the generators of $\widehat{A}(M-1, N-1)$ in the Cartan–Weyl basis with $\gamma = 1$ is given by

$$h_i^m = \sum_{r \in \mathbb{Z}'} \Big(: c_i^\dagger(r) c_i(r+m) : - : c_{i+1}^\dagger(r) c_{i+1}(r+m) : \Big) \qquad i = 1, \ldots, M-1 \tag{3.8a}$$

$$h_M^m = \sum_{r \in \mathbb{Z}'} \Big(: c_M^\dagger(r) c_M(r+m) : + : d_1^\dagger(r) d_1(r+m) : \Big) \tag{3.8b}$$

$$h_{M+k}^m = \sum_{r \in \mathbb{Z}'} \Big(: d_k^\dagger(r) d_k(r+m) : - : d_{k+1}^\dagger(r) d_{k+1}(r+m) : \Big) \qquad k = 1, \ldots, N-1 \tag{3.8c}$$

$$e_{\varepsilon_i - \varepsilon_j}^m = \sum_{r \in \mathbb{Z}'} c_i^\dagger(r) c_j(r+m) \tag{3.8d}$$

$$e_{\delta_k - \delta_l}^m = \sum_{r \in \mathbb{Z}'} d_k^\dagger(r) d_l(r+m) \tag{3.8e}$$

$$e_{\varepsilon_i - \delta_k}^m = \sum_{r \in \mathbb{Z}'} c_i^\dagger(r) d_k(r+m) \tag{3.8f}$$

$$e_{\delta_k - \varepsilon_i}^m = \sum_{r \in \mathbb{Z}'} d_k^\dagger(r) c_i(r+m). \tag{3.8g}$$

A fermionic oscillator realization of the simple generators of $\widehat{A}(M-1, N-1)$ in the distinguished Serre–Chevalley basis is given by ($\alpha = 0, 1, \ldots, R$)

$$h_\alpha = \sum_{r \in \mathbb{Z}'} h_\alpha(r) \qquad \text{and} \qquad e_\alpha^\pm = \sum_{r \in \mathbb{Z}'} e_\alpha^\pm(r) \tag{3.9}$$

where ($i = 1, \ldots, M-1$ and $k = 1, \ldots, N-1$)

$$h_i(r) = n_i(r) - n_{i+1}(r) =: n_i(r) : - : n_{i+1}(r) : \tag{3.10a}$$

$$h_M(r) = n_M(r) + n'_1(r) =: n_M(r) : + : n'_1(r) : \tag{3.10b}$$

$$h_{M+k}(r) = n'_k(r) - n'_{k+1}(r) =: n'_k(r) : - : n'_{k+1}(r) : \tag{3.10c}$$

$$h_0(r) = n'_N(r) + n_1(r+1) =: n'_N(r) : + : n_1(r+1) : - \delta_{r+1/2,0} \tag{3.10d}$$

$$e_i^+(r) = c_i^\dagger(r) c_{i+1}(r) \qquad e_i^-(r) = c_{i+1}^\dagger(r) c_i(r) \tag{3.10e}$$

$$e_M^+(r) = c_M^\dagger(r) d_1(r) \qquad e_M^-(r) = d_1^\dagger(r) c_M(r) \tag{3.10f}$$

$$e_{M+k}^+(r) = d_k^\dagger(r) d_{k+1}(r) \qquad e_{M+k}^-(r) = d_{k+1}^\dagger(r) d_k(r) \tag{3.10g}$$

$$e_0^+(r) = d_N^\dagger(r) c_1(r+1) \qquad e_0^-(r) = c_1^\dagger(r+1) d_N(r). \tag{3.10h}$$

Inserting equation (3.10*d*) into (3.9) and taking into account that the sum over r can be split into a sum of two convergent series only after normal ordering, one can check that

$$h_0 = -1 + \sum_{r\in\mathbb{Z}'} : n'_N(r) : + \sum_{r\in\mathbb{Z}'} : n_1(r) := -1 + \sum_{i=1}^{M-1} h_i^0 + h_M^0 - \sum_{k=1}^{N-1} h_{M+k}^0 \tag{3.11}$$

i.e. the central charge is $\gamma = 1$.

Note that the value of the central charge is related to the definition of the normal ordered product. A different definition like ($i = 1, \ldots, M$ and $k = 1, \ldots, N$)

$$: n_i(r) := n_i(r) \qquad \text{and} \qquad : n'_k(r) := n'_k(r) \qquad \text{for any } r \in \mathbb{Z}' \tag{3.12}$$

would lead to $\gamma = 0$.

4. Anyonic realization of $\mathcal{U}_q(\widehat{A}(M-1, N-1))$

In order to obtain an anyonic realization of $\mathcal{U}_q(\widehat{A}(M-1, N-1))$, we will replace the fermionic and bosonic oscillators by suitable anyons in the expressions of the simple generators of $\mathcal{U}_q(\widehat{A}(M-1, N-1))$ in the distinguished Serre–Chevalley basis. Since we have to deal with fermionic and bosonic generators, we have to introduce two different types of anyons.

Let us first define fermionic anyons on a one-dimensional lattice $\mathbb{Z}'$ [7, 5]:

$$a_i(r) = K_i(r) c_i(r) \qquad \text{and} \qquad \tilde{a}_i(r) = \tilde{K}_i(r) c_i(r) \tag{4.1}$$

and similar expressions for the conjugated operators $a_i^\dagger(r)$ and $\tilde{a}_i^\dagger(r)$, where the disorder factors $K_i(r)$ and $\tilde{K}_i(r)$ are expressed as

$$K_i(r) = q^{-\frac{1}{2}\sum_{t\in\mathbb{Z}'} \varepsilon(t-r) : n_i(t):} \tag{4.2a}$$

$$\tilde{K}_i(r) = q^{\frac{1}{2}\sum_{t\in\mathbb{Z}'} \varepsilon(t-r) : n_i(t):} . \tag{4.2b}$$

The function $\varepsilon(t) = |t|/t$ if $t \neq 0$ and $\varepsilon(0) = 0$ is the sign function.

It is easy to prove that the a-type anyons satisfy the following braiding relations for $r > s$:

$$\begin{aligned} &a_i(r) a_i(s) + q^{-1} a_i(s) a_i(r) = 0 \\ &a_i^\dagger(r) a_i^\dagger(s) + q^{-1} a_i^\dagger(s) a_i^\dagger(r) = 0 \\ &a_i^\dagger(r) a_i(s) + q\, a_i(s) a_i^\dagger(r) = 0 \\ &a_i(r) a_i^\dagger(s) + q\, a_i^\dagger(s) a_i(r) = 0 \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} &a_i(r) a_i^\dagger(r) + a_i^\dagger(r) a_i(r) = 1 \\ &a_i(r)^2 = a_i^\dagger(r)^2 = 0. \end{aligned} \tag{4.4}$$

The braiding relations between the $\tilde{a}$-type anyons are obtained from equations (4.3) and (4.4) by replacing $q \leftrightarrow q^{-1}$.

Finally, the braiding relations between a-type and $\tilde{a}$-type anyons are given by

$$\left\{\tilde{a}_i(r), a_i(s)\right\} = \left\{\tilde{a}_i^\dagger(r), a_i^\dagger(s)\right\} = 0 \qquad \text{for all } r, s \in \mathbb{Z}' \tag{4.5}$$

$$\left\{\tilde{a}_i^\dagger(r), a_i(s)\right\} = \left\{\tilde{a}_i(r), a_i^\dagger(s)\right\} = 0 \qquad \text{for all } r \neq s \in \mathbb{Z}' \tag{4.6}$$

and

$$\begin{aligned} \left\{\tilde{a}_i(r), a_i^\dagger(r)\right\} &= q^{\sum_{t\in\mathbb{Z}'} \varepsilon(t-r):n_i(t):} \\ \left\{\tilde{a}_i^\dagger(r), a_i(r)\right\} &= q^{-\sum_{t\in\mathbb{Z}'} \varepsilon(t-r):n_i(t):}. \end{aligned} \tag{4.7}$$

Moreover, the following identity holds:

$$a_i^\dagger(r)a_i(r) = \tilde{a}_i^\dagger(r)\tilde{a}_i(r) = n_i(r) \tag{4.8}$$

the normal ordering between a-type and $\tilde{a}$-type anyons being defined as in (3.4).

Now we will define anyonic-like operators based on q-deformed bosons. Let us recall that q-deformed bosons can be constructed from ordinary ones by the following procedure [12]:

$$n'_k(r) = d_k^\dagger(r)d_k(r) \tag{4.9a}$$

$$b_k(r) = d_k(r)\sqrt{\frac{[n'_k(r)]_q}{n'_k(r)}} = \sqrt{\frac{[n'_k(r)+1]_q}{n'_k(r)+1}}\, d_k(r) \tag{4.9b}$$

$$b_k^\dagger(r) = \sqrt{\frac{[n'_k(r)]_q}{n'_k(r)}}\, d_k^\dagger(r) = d_k^\dagger(r)\sqrt{\frac{[n'_k(r)+1]_q}{n'_k(r)+1}}. \tag{4.9c}$$

The q-deformed bosons $b_k(r), b_k^\dagger(r)$ satisfy the following q-commutation relations:

$$b_k(r)b_l^\dagger(s) - q^{\delta_{kl}\delta_{rs}} b_l^\dagger(s)b_k(r) = q^{-n'_k(r)}\delta_{kl}\delta_{rs} \tag{4.10a}$$

$$b_k(r)b_l^\dagger(s) - q^{-\delta_{kl}\delta_{rs}} b_l^\dagger(s)b_k(r) = q^{n'_k(r)}\delta_{kl}\delta_{rs} \tag{4.10b}$$

$$b_k(r)b_l(s) - b_l(s)b_k(r) = b_k^\dagger(r)b_l^\dagger(s) - b_l^\dagger(s)b_k^\dagger(r) = 0 \tag{4.10c}$$

$$\left[n'_k(r), b_l(s)\right] = -b_k(r)\delta_{kl}\delta_{rs} \tag{4.10d}$$

$$\left[n'_k(r), b_l^\dagger(s)\right] = b_k^\dagger(r)\delta_{kl}\delta_{rs} \tag{4.10e}$$

from which it follows that

$$b_k^\dagger(r)b_k(r) = [n'_k(r)]_q \qquad \text{and} \qquad b_k(r)b_k^\dagger(r) = [n'_k(r)+1]_q. \tag{4.11}$$

Now, let us define anyonic-like operators as follows:

$$A_k(r) = K'_k(r)b_k(r) \qquad \text{and} \qquad \tilde{A}_k(r) = \tilde{K}'_k(r)b_k(r) \tag{4.12}$$

and similar expressions for the conjugate operators $A_k^\dagger(r)$ and $\tilde{A}_k^\dagger(r)$, where the disorder factors are given by

$$K'_k(r) = q^{\frac{1}{2}\sum_{t\in\mathbb{Z}'} \varepsilon(t-r):n'_k(t):} \tag{4.13a}$$

$$\tilde{K}'_k(r) = q^{-\frac{1}{2}\sum_{t\in\mathbb{Z}'} \varepsilon(t-r):n'_k(t):}\ . \tag{4.13b}$$

It can be proved that the operators $A_k(r)$, $A_k^\dagger(r)$ satisfy the following braiding relations for $r > s$:

$$\begin{aligned}
&A_k(r)A_k(s) - qA_k(s)A_k(r) = 0\\
&A_k^\dagger(r)A_k^\dagger(s) - qA_k^\dagger(s)A_k^\dagger(r) = 0\\
&A_i^\dagger(r)A_k(s) - q^{-1}A_k(s)A_i^\dagger(r) = 0\\
&A_k(r)A_k^\dagger(s) - q^{-1}A_k^\dagger(s)A_k(r) = 0
\end{aligned} \tag{4.14}$$

and

$$\begin{aligned}
&A_k(r)A_k^\dagger(r) - qA_k^\dagger(r)A_k(r) = q^{-n'_k(r)}\\
&A_k(r)A_k^\dagger(r) - q^{-1}A_k^\dagger(r)A_k(r) = q^{n'_k(r)}.
\end{aligned} \tag{4.15}$$

Therefore, the operators $A_k(r)$, $A_k^\dagger(r)$ satisfy the q-commutation relations of the q-deformed bosonic oscillator at the same point, while they satisfy braiding relations when taken at different points.

The braiding relations between the $\tilde{A}$-type anyons are obtained from (4.14) by replacing $q \leftrightarrow q^{-1}$.

Note that for these anyonic q-deformed bosons defined by the above relations, we do not have any physical interpretations, on the contrary of the a-type anyons, see [7, 5]. Is is worth to point out that the above introduced *bosonic anyons* differ from the ones introduced in [9] by the non-trivial fact that our anyons are defined on a lattice while in [9] are defined in the continuum and by the local braiding relation (4.15). Replacing the q-boson in (4.12) by a standard boson, we find in the lattice the bosonic anyons of [9]. We will come back on the difference between the two approaches in the next section.

Now we can build an anyonic realization of $\mathcal{U}_q(\widehat{A}(M-1,N-1))$ by 'anyonizing' the oscillator realization (3.10*a*)–(3.10*h*), i.e. replacing the fermionic oscillators c_i by the anyonic oscillators a_i and $\tilde{a}_i$ and the bosonic oscillators b_i by the operators A_i and $\tilde{A}_i$. More precisely, one has:

Theorem 1. An anyonic realization of the simple generators of the quantum affine Lie superalgebra $\mathcal{U}_q(\widehat{A}(M-1,N-1))$ with central charge $\gamma = 1$ is given by (with $\alpha = 0, 1, \ldots, R$)

$$H_\alpha = \sum_{r\in\mathbb{Z}'} H_\alpha(r) \qquad \text{and} \qquad E_\alpha^\pm = \sum_{r\in\mathbb{Z}'} E_\alpha^\pm(r) \tag{4.16}$$

where ($i = 1, \ldots, M-1$ and $k = 1, \ldots, N-1$)

$$H_i(r) = n_i(r) - n_{i+1}(r) =: n_i(r) : - : n_{i+1}(r) : \tag{4.17a}$$

$$H_M(r) = n_M(r) + n'_1(r) =: n_M(r) : + : n'_1(r) : \tag{4.17b}$$

$$H_{M+k}(r) = n'_k(r) - n'_{k+1}(r) =: n'_k(r) : - : n'_{k+1}(r) : \tag{4.17c}$$

$$H_0(r) = n'_N(r) + n_1(r+1) =: n'_N(r) : + : n_1(r+1) : -\delta_{r+1/2,0} \tag{4.17d}$$

$$E_i^+(r) = a_i^\dagger(r)a_{i+1}(r) \qquad E_i^-(r) = \tilde{a}_{i+1}^\dagger(r)\tilde{a}_i(r) \tag{4.17e}$$

$$E_M^+(r) = a_M^\dagger(r)A_1(r) \qquad E_M^-(r) = \tilde{A}_1^\dagger(r)\tilde{a}_M(r) \tag{4.17f}$$

$$E_{M+k}^+(r) = A_k^\dagger(r)A_{k+1}(r) \qquad E_{M+k}^-(r) = \tilde{A}_{k+1}^\dagger(r)\tilde{A}_k(r) \tag{4.17g}$$

$$E_0^+(r) = A_N^\dagger(r)a_1(r+1) \qquad E_0^-(r) = \tilde{a}_1^\dagger(r+1)\tilde{A}_N(r). \tag{4.17h}$$

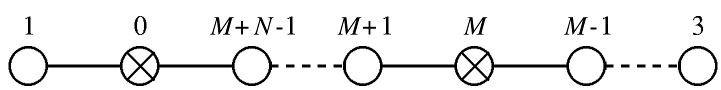
Proof. We must check that the realization (4.16) and (4.17*a*)–(4.17*h*) indeed satisfy the quantum affine Lie superalgebra $\mathcal{U}_q(\widehat{A}(M-1,N-1))$ in the distinguished Serre–Chevalley basis (2.8*a*), (2.8*d*), together with the quantum Serre relations (2.9) and (2.11). The proof follows the lines of the algebraic case [3]: equations (2.8*a*)–(2.11) which define a generic deformed affine superalgebra $\mathcal{U}_q(\widehat{\mathcal{A}})$ reduce to $\mathcal{U}_q(\mathcal{A})$ when the affine dot is removed and to another finite dimensional superalgebra $\mathcal{U}_q(\mathcal{A}')$ if the affine dot is kept and one or more other suitable dots are removed. The relations defining $\mathcal{U}_q(\widehat{\mathcal{A}})$ coincide with the union of those defining $\mathcal{U}_q(\mathcal{A})$ and $\mathcal{U}_q(\mathcal{A}')$: therefore, it will be enough to check that the equations defining $\mathcal{U}_q(\mathcal{A})$ and $\mathcal{U}_q(\mathcal{A}')$ are satisfied.

Consider the non-extended Dynkin diagram of $A(M-1,N-1)$ to which the set of generators $\{H_\alpha, E_\alpha^\pm\}$ (with $\alpha \neq 0$) corresponds. Inserting equations (4.1), (4.2*a*), (4.2*b*), (4.11), (4.12), expressions (4.17*e*)–(4.17*g*) become

$$E_\alpha^\pm(r) = \hat{e}_\alpha^\pm(r)\, q_\alpha^{\frac{1}{2}\sum_{t\in\mathbb{Z}'}\varepsilon(t-r)\,:h_\alpha(t):} \tag{4.18}$$

where the generators $\hat{e}_\alpha^\pm(r)$ are obtained from the generators $e_\alpha^\pm(r)$ in (3.10*e*)–(3.10*g*) replacing the bosonic oscillators d_k by the q-deformed bosons b_k. The generators $\hat{e}_\alpha^\pm(r)$ coincide locally, i.e. for fixed r, with the generators of $\mathcal{U}_q(A(M-1,N-1))$ of [1, 2] as the q-deformed fermions ψ_i of [2] are equivalent to the usual fermionic oscillators c_i. It follows that the generators $\{H_\alpha, E_\alpha^\pm\}$ of (4.16) are a representation of $\mathcal{U}_q(A(M-1,N-1))$, since they are obtained with the correct coproduct (see equations (2.12*a*) and (4.18)) by the representation in terms of $\{h_\alpha, \hat{e}_\alpha^\pm\}$. Let us remark that due to the equivalence betwen q-fermions and standard fermions the realization of finite q-superalgebras of [2] are realizations of deformed algebras only for the subalgebra realized in terms of q-bosons while the subalgebra realized in terms of q-fermions is left undeformed. In contrast, the anyonic realization presented here is completely deformed in any sector. Finally let us remark that the difference $q \to q^{-1}$ in the disorder factor of the A-type anyons (in the site r) with respect to the a-type anyons (in the same site), see the q_α-factor in (4.18), is essential for the consistency of the q-superalgebra structure.

We consider then the *extended* Dynkin diagram of $A(M-1,N-1)$ and we delete a dot which is not the affine dot. For example, cutting the dot number 2, we obtain the following Dynkin diagram:



which corresponds to the Lie superalgebra $A(M-1,N-1)$ in a *non-distinguished* basis.

For a fixed $r \in \mathbb{Z}'$, it is possible to show that the set $\{h_j(r), \hat{e}_j^\pm(r), h_1(r+1), \hat{e}_1^\pm(r+1)\}$ ($j = 0, 3, \ldots, M+N-1$) is a representation of $\mathcal{U}_q(A(M-1,N-1))$ in the non-distinguished basis specified by the above Dynkin diagram. We emphasize that in this case we have to satisfy two more supplementary Serre relations than in the distinguished basis. Of course, for particular values of M and N one or both relations can be absent. Note that deleting the Mth dot, we recover the superalgebra $\mathcal{U}_q(A(M-1,N-1))$ in the distinguished basis. Then it follows that the generators $\{H_\alpha, E_\alpha^\pm\}$ with $\alpha \neq 2$ are a representation of $\mathcal{U}_q(A(M-1,N-1))$, as they are obtained by the generators of a representation of the finite q-superalgebra with the correct coproduct. This completes the proof. □

5. General representations and conclusions

In the previous section, we have built a representation of the deformed affine Lie superalgebras $\mathcal{U}_q(\widehat{A}(M-1,N-1))$ by means of anyons defined on an infinite linear chain;

as the corresponding fermionic representation, it has central charge $\gamma = 1$. Representations with vanishing central charge could be built in the same way by using alternative normal ordering prescriptions (3.12).

Representations with $\gamma = 0$ and $\gamma = 1$ can be combined together to get representations with arbitrary positive integer central charges (we do not discuss here the problem of the irreducibility of these representations). Associating a representation with any horizontal line of a two-dimensional square lattice, infinite in one direction (say the horizontal one), and taking K copies of representations in a one-dimensional lattice with central charge equal to 1, one can obtain representations with the value of the central charge equal to K. Note that by combining one representation with central charge equal to K with a finite number of representations (in one-dimensional lattice) with vanishing value of the central charge one obtains an inequivalent representation with the same value of the central charge. The extension to a two-dimensional lattice infinite in both directions can also be done, but it requires some care in the definition in order to avoid convergence problems.

We have shown in [3] that the use of a-anyons on a two-dimensional lattice naturally gives the correct coproduct with the correct powers of the deformation of the representations of a q-algebra defined in a fixed site of the lattice. For completeness we recall that each site of the two-dimensional lattice is labelled by a vector $\boldsymbol{x} = (x_1, x_2)$, the first component $x_1 \in \mathbb{Z}'$ being the coordinate of a site on the line $x_2 \in \mathbb{Z}$. The angle $\Theta(\boldsymbol{x}, \boldsymbol{y})$ which enters into the definition of two-dimensional a-anyons through the disorder factor (see, e.g., [5])

$$K(\boldsymbol{x}) = \exp\left(\mathrm{i}\nu \sum_{\boldsymbol{y} \neq \boldsymbol{x}} \Theta(\boldsymbol{x}, \boldsymbol{y})\, n(\boldsymbol{y})\right) \tag{5.1}$$

may be chosen in such a way that

$$\Theta(\boldsymbol{x}, \boldsymbol{y}) = \begin{cases} +\pi/2 & \text{if } x_2 > y_2 \\ -\pi/2 & \text{if } x_2 < y_2 \end{cases} \tag{5.2}$$

while if $\boldsymbol{x}$ and $\boldsymbol{y}$ lie on the same horizontal line, i.e. $x_2 = y_2$, the definitions of section 4 hold. Two-dimensional anyons still satisfy the braiding and anticommutation relations expressed in general form in (4.3)–(4.7). Analogous relations hold for the A-anyons.

Let us replace the one-dimensional anyons in the equations of section 4 with two-dimensional ones and sum over the sites of the two-dimensional lattice. This sum has to be read as a sum over the infinite line x_1 and a sum over the finite number of lines labelled by x_2. Then the generators are given by a sum, with the correct coproduct, of the generators of a $\mathcal{U}_q(\widehat{A}(M-1, N-1))$ representations defined in a line. Therefore they define a $\mathcal{U}_q(\widehat{A}(M-1, N-1))$ representation with value of the central charge given by the sum of the values (0 or 1, see the discussion in section 4) of the central charges associated with each line of the two-dimensional lattice.

In the previous sections we have discussed the case of $|q| = 1$. The case of q real can also be discussed and we refer the reader to [7] for the definition of anyons for generic q.

One can naturally ask if the realization in terms of a-anyons and A-anyons presented here can be used to realize the deformation of other finite or affine superalgebras. It seems that this procedure can be extended to the other basic finite superalgebras, i.e. the series $B(0, N)$, $B(M, N)$, $C(N+1)$, $D(M, N)$, while it is not clear whether it can be extended to the exceptional or strange finite superalgebras or to the affine case. Finally, we want briefly to comment on the difference between our approach and that used in [9], even if here we present the realization of a q-superalgebra and in [9] a realization of a q-algebra was presented. The approach of [9] is made on the continuum and, as already remarked, the authors do not use q-bosons as in the present paper, but standard bosons before ‘anyonization’. However, one has to stress that their approach is based on a Fock space realization which guarantees the consistency of the commutation relations. It is worth noting that on the Fock space the fundamental representation of the deformed algebra is indistinguishable from the fundamental representation of the undeformed algebra. It follows that a sum with the correct product of the fundamental representation gives a representation of the deformed algebra. In contrast, as we fulfill the commutation relations in abstract way, we are not allowed to consider only the fundamental representation of the deformed algebra realized by bosons, and in order to achieve the consistency of the representation we are lead to use q-bosons.

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